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VARIATIONAL APPROACH FOR PLANE PROBLEMS OF MATCHING POTENTIAL AND VORTICAL FLOWS

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A direct variational method is used to show that a fluid flow with a continuous velocity field exists in a bounded region, which is potential in a part of the region and has a specified vorticity in the remaining part. The basic difficulty is to prove the smoothness of a solution which would satisfy the Euler equation with discontinuous coefficients.

In a number of cases when the motion of the fluid is studied at large Reynolds numbers, problems arise of matching smoothly a potential and a vortical flow (see e.g. [1 — 3]). In the plane case the problems are formulated as follows: to determine in a specified region Ω a continuously differentiable stream function ψ and a curve γ so that $\psi = 0$ on γ and $\Delta\psi = 0$ on the side of γ on which $\psi > 0$, while on the side where $\psi < 0$, the Laplacian $\Delta\psi$ assumes a prescribed value of the vorticity $\omega(x)$. The boundary value conditions for ψ are determined from the hydrodynamic considerations, and the region Ω is not necessarily bounded. Other similar formulations of the problem are also possible.

Problems of matching the flows were studied numerically and analytically by many authors using the above formulation. Certain properties of the line γ have been established in [5 — 8] and the existence of a solution was proved in [9] for the case when the region Ω is bounded. All the above papers used the method of reducing the problem to a certain nonlinear integral equation and solving the latter numerically. In [9] it was mentioned that the problem of matching allows a variational formulation. Below we use a direct variational method to solve that problem in bounded regions. The proof does not depend on the dimensionality n of the space, and is therefore given in its most general form. Naturally, the hydrodynamic interpretation is possible only when $n = 2$. The solution of the problem is represented by an extremum of a certain discontinuous functional I which is approached, as $\varepsilon \to 0$, by a family of smooth functionals I_ε; in this way the smoothness of the solution is proved. An interesting problem on the number of solutions remains open. The standard notation is used throughout this paper (see [10], Ch. Sect. 1).

Let $\Omega \subset E_n$, $n \geqslant 2$ be a region with a sufficiently boundary (e.g. a piecewise smooth boundary of class C^∞ with nonzero angles), ω and ε be some positive numbers, and $\varphi \in C^\infty(\Omega)$. We define in $W_2^1(\Omega)$ the functional

$$I_\varepsilon(\psi) = \int_\Omega [\,|\nabla\psi|^2 + \omega(\psi - \sqrt{\varepsilon + \psi^2})]\,dx$$

and consider the variational problem $P(\varepsilon)$: to minimize $I_\varepsilon(\psi)$ in the class of all functions $\psi \in \varphi + \overset{\circ}{W}{}_2^{1}(\Omega)$.

T h e o r e m 1. The problem $P(\varepsilon)$ has at least one solution; for any solution ψ_ε of this problem the following estimate uniform in $\varepsilon \in (0, 1]$ holds:

$$\max_\Omega |\psi_\varepsilon| \leqslant C(\Omega, \varphi, \omega) \tag{1}$$

Proof. First we confirm the validity of the standard lemmas on which the direct method of variational computation is based.

Lemma 1. The values of the functional I on the set $\varphi + \overset{\circ}{W}{}_2^1(\Omega)$ are uniformly bounded from below.

Lemma 2. If $I_\varepsilon(\psi) < C'$, $\psi \in \varphi + \overset{\circ}{W}{}_2^1(\Omega)$, then

$$\|\psi\|_{W_2^1(\Omega)} < C(\Omega, \omega, \varphi, C')$$

The above lemmas can be proved using the Schwartz and Poincaré inequalities.

Lemma 3. The functional I_ε is semicontinuous from below relative to the weak convergence in $W_2^1(\Omega)$ on the set $\varphi + \overset{\circ}{W}{}_2^1(\Omega)$.

Assuming that

$$I_\varepsilon = I_{\varepsilon 1} + I_{\varepsilon 2} - I_{\varepsilon 3}$$

$$I_{\varepsilon 1} = \int_\Omega |\nabla\psi|^2 dx, \quad I_{\varepsilon 2} = \omega\int_\Omega \psi\, dx, \quad I_{\varepsilon 3} = \omega\int_\Omega \sqrt{\varepsilon + \psi^2} dx$$

we see that the semicontinuity of the functional $I_{\varepsilon 1}$ from below is certain, the functional $I_{\varepsilon 2}$ is linear and $I_{\varepsilon 3}$ is continuous relative to the weak convergence on the set $\varphi + \overset{\circ}{W}{}_2^1(\Omega)$. Indeed, if $\psi_k \to \psi$ is weak in $\varphi + \overset{\circ}{W}{}_2^1(\Omega)$, then the Rellich theorem guarantees the strong convergence $\psi_k \to \psi$ in $L_2(\Omega)$ and evidently also in $L_1(\Omega)$. It remains to take into account the inequality

$$|\sqrt{\varepsilon + \psi_k^2} - \sqrt{\varepsilon + \psi^2}| \leqslant |\psi_k - \psi|$$

and the proof of existence of at least one solution of the problem $P(\varepsilon)$ can now be carried out using the standard methods (see e.g. [11]).

Next we prove that every solution ψ_ε of the problem $P(\varepsilon)$ is bounded. It is evident, that $\operatorname{vrai}\sup_\Omega \psi \leqslant \sup_{\partial\Omega} \varphi$. Indeed, if it were not so, then for every $k > \sup_{\partial\Omega}\varphi$ for the section $\psi^{(k)}(x) = \min\{\psi(x), k\} \in \varphi + \overset{\circ}{W}{}_2^1(\Omega)$ we would have $I_\varepsilon(\psi^{(k)}) < I_\varepsilon(\psi)$. The proof that the solution is bounded from below by a constant independent of ε, is somewhat more involved, and here we can use an argument analogous to that given at the end of Sect. 2 in [12].

Corollary [13]. Any solution ψ_ε of the variational problem $P(\varepsilon)$ is a classical solution of the Euler equation for the functional I_ε

$$\Delta\psi = F(\omega, \psi, \varepsilon), \quad F(\omega, \psi, \varepsilon) = {}^1/_2\, \omega\,(1 - \psi / \sqrt{\varepsilon + \psi^2}), \ \psi|_{\partial\Omega} = \varphi \tag{2}$$

In particular, the solution ψ_ε is analytic at all internal points of the region Ω.

Now we turn our attention to the basic variational problem P: to minimize, on the set of functions $\psi \in \varphi + \overset{\circ}{W}{}_2^1(\Omega)$, the functional

$$I(\psi) = \int_\Omega (|\nabla\psi|^2 + 2\omega \min\{\psi(x), 0\})\, dx$$

Theorem 2. The problem P has at least one solution $\psi \in C^{1+\alpha}(\overline{\Omega})$, $\alpha = \alpha(\omega) \in (0, 1]$; the function ψ is analytic everywhere in $\Omega \setminus \{x \in \Omega : \psi(x) = 0\}$ and $\Delta\psi(x) = 0$, when $\psi(x) > 0$ and $\Delta\psi(x) = \omega$, when $\psi(x) < 0$.

Proof. Any solution ψ_ε of the problem $P(\varepsilon)$ will satisfy Eq. (2) with the right-hand side admitting the estimate $|F(\omega, \psi, \varepsilon)| \leqslant \omega$ independent of ε. Therefore, using the estimate (1) and Theorem (6.5) of ([10], ch. 4), we can find a number $\alpha \in (0,1]$

independent of ε and such that

$$\|\psi_\varepsilon\|_{C^{1+\alpha}(\bar\Omega)} \leqslant C(\Omega, \varphi, \omega)$$

But in this case the function $\psi_k = \psi_{\varepsilon_k}$ will converge in the norm $C^{1+\alpha/2}(\bar\Omega)$ for some sequence $\varepsilon_k \to 0$ to some function $\psi \in \{\varphi + W_2^{\circ 1}(\Omega)\} \cap C^{1+\alpha}(\bar\Omega)$. We can now prove, as in [12], that ψ is a solution of the problem P.

Setting now $\delta > 0$, let us consider the set $\Omega_\delta = \{x \in \Omega : |\psi(x)| > \delta\}$. From some value of k onwards, $|\psi_k(x)| > \delta/2$, provided that $x \in \Omega_\delta$. In particular, the following estimate holds on Ω_δ:

$$\|F(\omega, \psi_k, \varepsilon)\|_{C^\alpha(\bar\Omega_\delta)} < C(\delta, \Omega, \omega, \varphi)$$

Applying the inner Schauder estimates to any subregion $\Omega_\delta' \subset\subset \Omega_\delta$, we find, that the norms $\|\psi_k\|_{C^{2+\alpha}(\bar\Omega_\delta')}$ are uniformly bounded.

Choosing now a suitably corresponding subsequence, we can assume that $\|\psi_k - \psi\| \to 0$ in the norm $C^{2+\alpha/2}(\bar\Omega_\delta')$ and, that $\psi \in C^{2+\alpha}(\bar\Omega_\delta')$. In particular, for any $x \in \Omega_\delta'$ we have

$$\Delta\psi(x) = \lim_{k\to\infty} \Delta\psi_k(x) = \lim_{k\to\infty} F(\omega, \psi_k(x), \varepsilon_k) = \begin{cases} 0, & \psi_k > 0 \\ \omega, & \psi_k < 0 \end{cases}$$

Since we can exhaust all $\Omega \setminus \{x \in \Omega : \psi(x) = 0\}$, with sets Ω_δ', Theorem 2 is proved.

Notes 1°. At sufficiently large ω the set $\{x \in \Omega : \psi(x) < 0\}$ is nonempty for any solution ψ of the problem P. Otherwise for any $\omega > 0$ the solution ψ would be a harmonic function assuming prescribed values at the boundary $\partial\Omega$, i.e. it would be independent of ω. But in this case for the arbitrary function $\psi_0 \in C^2(\Omega) \cap \{\varphi + W_2^{\circ 1}(\Omega)\}$, such that $\psi_0 < 0$ somewhere in Ω, we can find ω so large that the inequality $I(\psi_0) < I(\psi)$ will hold.

2°. Theorems 1 and 2 can be directly generalized to functionals with sufficiently smooth $\omega = \dot\omega(x)$.

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REFERENCES

1. Batchellor, G. K., On steady laminar flow with closed streamlines at large Reynolds number. J. Fluid Mech., Vol. 1, pt. 2, 1956.

2. Batchellor, G. K., A proposal concerning laminar wakes behind bluff bodies at large Reynolds number. J. Fluid Mech., Vol. 1, pt. 4, 1956.

3. Lavrent'ev, M. A. and Shabat, B. V., Problems of Hydrodynamics and their Mathematical Models. Moscow, "Nauka", 1973.

4. Vainshtein, I. I. and Gol'dshtik, M. A., On the motion of a perfect fluid in a Coriolis force field. Dokl. Akad. Nauk SSSR, Vol. 173, № 6, 1967.

5. Shabat, A. B., On a pattern of plane flow of fluid in the presence of a trench at the bottom. PMTF, № 4, 1962.

6. Shabat, A. B., On two problems of matching. Dokl. Akad. Nauk SSSR, Vol. 150, № 6, 1963.

7. Sadovskii, V. S., Region of constant vorticity in a plane potential flow. Uch. zap. TsAGI, Vol. 1, № 4, 1970.

8. Sadovskii, V. S., On certain properties of potential and vortical flow contiguous along a closed fluid streamline. Uch. zap. TsAGI, Vol. 2, № 1, 1971.

9. Gol'dshtik, M. A., Mathematical model of detached flows of incompressible fluid. Dokl. Akad. Nauk SSSR, Vol. 147, № 6, 1962.

10. Ladyzhenskaia, O. A. and Ural'tseva, N. N., Linear and Quasilinear Elliptic Equations. Moscow, "Nauka", 1973.

11. Morrey, C. B., Multiple integrals in the calculus of variations. Berlin, Springer-Verlag, 1966.

12. Titov, O. V., Minimal Hypersurfaces Under Mild Obstructions. Izv. Akad. Nauk SSSR, Ser. matem., Vol. 38, № 2, 1974.

13. Stampacchia, G., On some regular multiple integral problems in the calculus of variations. Commun. Pure and Appl. Math., Vol. 16, № 4, 1963.

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